

Algorithms for Unevenly-Spaced Time Series: Moving Averages and Other Rolling Operators

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Abstract

This paper describes algorithms for efficiently calculating certain rolling time series operators for unevenly-spaced data. In particular, we show how to calculate simple moving averages (SMAs), exponential moving averages (EMAs), and related operators in $O(n)$ time. A web appendix provides an implementation of these algorithms in the programming language C and a package (forthcoming) for the statistical software R.¹

Keywords: unevenly spaced time series, unequally spaced time series, irregularly spaced time series, moving average, simple moving average, exponential moving average, exponentially weighted moving average, R

1 Introduction

There exists an extensive body of literature on analyzing *equally*-spaced time series data, see [Tong \(1990\)](#), [Brockwell and Davis \(1991\)](#), [Hamilton \(1994\)](#), [Brockwell and Davis \(2002\)](#), [Fan and Yao \(2003\)](#), [Box et al. \(2004\)](#), and [Lütkepohl \(2010\)](#). As a consequence, efficient algorithms have been developed for many computational questions, and implementations are available for software environments such as Fortran, Matlab[®], R, and SAS[®]. Many of the basic algorithms were developed at a time when limitations in computing resources favored an analysis of equally-spaced data and the use of linear Gaussian models, since efficient linear algebra routines can be used in this case.

As a consequence, little literature exists specifically on algorithms for analyzing and processing *unevenly*-spaced (also called *unequally*- or *irregularly*-spaced) time series data, even though such data naturally occurs in many industrial and scientific domains, such as astronomy, biology, climatology, economics, engineering, geology, and signal processing.

For such data, [Müller \(1991\)](#) and [Dacorogna et al. \(2001\)](#) recommend, due to the computational simplicity, to use exponential moving averages (EMAs) as the main building block of time series operators. In particular, [Müller \(1991\)](#) argues that the sequential computation of EMAs “is more efficient than the computation of any differently weighted MA.” This paper shows that many other time series operators can in fact be calculated just as efficiently.

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¹See www.eckner.com/research.html.

The main focus of this paper are rolling time series operators, such as moving averages, that allow to extract a certain piece of local information about a time series (typically within a rolling window of a fixed length $\tau > 0$). The generic structure for most such operators is as follows:

```

for each window of length tau with end point equal to an observation time (
  do some calculation on
    the observation values and times in the current rolling window
)

```

Generic Rolling Time Series Operator

If the calculation for each window runs in linear time with respect to the number of observations in the said window, then the total time for applying such a rolling time series operator is proportional to (i) the length of the time series, times (ii) the average number of observations in each window (or equivalently, the average observation density divided by the windows length τ). However, in many cases it is possible to devise a much more efficient algorithm, with execution time proportional only to the length of the time series. This paper presents such algorithms for SMAs, EMAs, and various other rolling time series operators.

1.1 Basic Framework

This section provides a brief introduction to unevenly-spaced time series and the basic structure of rolling time series operators for such objects. For a much more detailed exposition see [Eckner \(2011\)](#).

We use the notation $((t_n, X_n) : 1 \leq n \leq N(X))$ or $(X_{t_n} : 1 \leq n \leq N(X))$ to denote an unevenly-spaced time series X with observation times $T(X) = \{t_1, \dots, t_{N(X)}\}$ and observation values $V(X) = (X_1, \dots, X_{N(X)})$, where $N(X)$ denotes the length of the time series. For a time point $t \in \mathbb{R}$, $X[t]$ denotes the most recent observation value of X at or before time t , while $X[t]_{\text{lin}}$ denotes the linearly interpolated value of X at time t . We call these quantities the sampled value X at time t with last-point and linear interpolation, respectively. For time points t before the first observation time of X , that is for $t < \min T(X)$, we define the sampled value as equal to the first observation value of X . While potentially not appropriate for some applications, this convention avoids the treatment of a multitude of special cases in the exposition below.²

The algorithms in this paper have as input (i) an array `values` containing the observation values, (ii) an array `times` containing the observation times, and (iii) a parameter τ describing the temporal horizon of a time series operator, such as the length of a moving average window. The output is an array `out` of same length as the two input arrays. Indices of arrays start at one. For integers $n \leq m$, $n : m$ denotes the array of numbers $[n, n + 1, \dots, m - 1, m]$. For brevity of the presentation (but not in the accompanying implementation) we ignore memory allocation, numerical noise, and special cases for time series of length zero or one.

Many of the algorithms below use a half-open rolling time window of the form $(t - \tau, t]$ to keep track of observations relevant to the calculation of a time series operator at a time $t \in T(X)$. Specifically, the variable `right` denotes the index corresponding to the right edge of the rolling time window, while the variable `left` denotes the index of the left-most observation

²A software implementation might instead use a special symbol to denote a value that is not available. For example, R uses the constant `NA`, which propagates through all steps of an analysis since the result of any calculation involving `NA`s is also `NA`.

inside the rolling time window. In other words,

$$t - \tau < \text{times}[\text{left}] \leq \text{times}[\text{right}] = t.$$

With this notation, the generic algorithm for rolling time series operators becomes:

```

left = 1;
for (right in 1:N(X)) {
  // Shrink window on the left end
  while (times[left] <= times[right] - tau)
    left = left + 1;

  // Calculate output for current rolling window
  out[right] = do_some_calculation(values[left:right], times[left:right]);
}

```

Generic Algorithm for Rolling Time Series Operators

Here, `values[left:right]` and `times[left:right]` denotes the array of observation values and times, respectively, in the current rolling time window.

In many cases of interest, the function `do_some_calculation` will run in linear time with respect to the number of observations inside the current time window. The algorithms in this paper improve the efficiency of this generic algorithm by reusing results from the previous time window to determine the output value of the current time window.

Remark 1. *In some cases, it is even desirable to store every single observation value within the rolling time window in a self-balancing binary search tree, such as an AVL tree, see [Knuth \(1998\)](#). Such a data structure allows searches, insertions, and deletions to be carried on in logarithmic time. However, the extra complexity involved in the implements pays off only for extremely large datasets. For time series of moderate length, a simpler algorithm is often just as efficient.*

2 Simple Moving Averages

Definition 1. *For an unevenly-spaced time series X , we define three versions of the simple moving average (SMA) of length $\tau > 0$. For $t \in T(X)$,*

- (i) $\text{SMA}(X, \tau)_t = \frac{1}{\tau} \int_0^\tau X[t-s] ds,$
- (ii) $\text{SMA}_{\text{lin}}(X, \tau)_t = \frac{1}{\tau} \int_0^\tau X[t-s]_{\text{lin}} ds,$
- (iii) $\text{SMA}_{\text{eq}}(X, \tau)_t = \text{avg}\{X_s : s \in T(X) \cap (t-\tau, t]\}$

where in all cases the observation times of the input and output time series are identical.

The first SMA uses last-point sampling, the second SMA uses linear interpolation, and the last SMA corresponds to the most widely used version for equally-spaced data. See [Eckner \(2011\)](#), Section 7.1 for a theoretical analysis of these moving averages for unevenly-spaced data.

The first SMA seems most appropriate for analyzing discrete observation *values*, for example, for calculating the average FED funds target rate³ over the past three years. In this case,

³The FED funds target rate is the desired interest rate (by the Federal Reserve) at which depository institutions (such as a savings bank) lend balances at the Federal Reserve to other depository institutions overnight. See www.federalreserve.gov/fomc/fundsrate.htm for details.

it is desirable to weigh observations by the amount of time each value remained unchanged. The SMA_{eq} is most appropriate for analyzing discrete *events*, for example, for calculating the average number of casualties per deadly car accident over the past twelve months, or for determining the average number of IBM common shares traded on the NYSE per executed order during the past 30 minutes. The SMA_{lin} can be used to estimate the rolling average value of certain discretely observed continuous time stochastic processes if the observation times are independent of the observation values, see Eckner (2011), Theorem 5.10.

2.1 SMA_{eq}

The simple moving average SMA_{eq} can be calculated efficiently by keeping track of (i) the number and (ii) sum of observation values in a window of length τ that moves forward in time. Both values are updated whenever a new observation enters or leaves the time window. The pseudocode of the algorithm is as follows:

```

left = 1; roll_sum = 0;
for (right in 1:N(X)) {
  // Expand window on right
  roll_sum = roll_sum + X[right];

  // Shrink window on left
  while (times[left] <= times[right] - tau) {
    roll_sum = roll_sum - X[left];
    left = left + 1;
  }

  // Save SMA value for current time window
  out[right] = roll_sum / (right - left + 1);
}

```

SMA_{eq} Algorithm

For very large datasets, to avoid numerical noise due to a potentially large number of additions and subtractions, it is recommended to occasionally calculate the sum of observation values in the current window from scratch.

2.2 SMA

The simple moving average SMA can be efficiently calculated in a manner similar to the SMA_{eq} . However, now the spacing of observations leaving and entering the rolling time window needs to be taken into account.

Figure 1 visualizes two key steps in the algorithm, which is as follows:

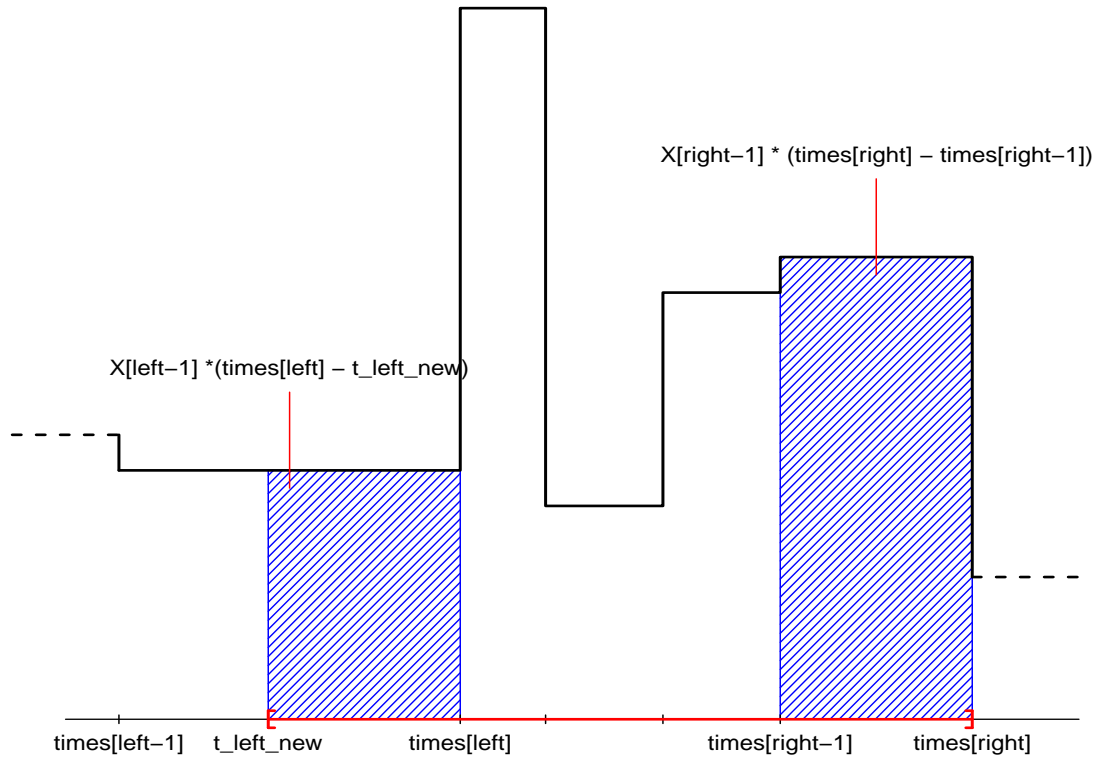


Figure 1: Areas involved in the calculation of the simple moving average $\text{SMA}(X, \tau)$. The step function is the sample path of a time series X with last-point sampling scheme. The shaded area on the right-hand side has to be added to the variable `roll_area` in order to expand the rolling time window to the right. Depending on the position in the algorithm, the shaded area on the left-hand side has to be either added or subtracted from the variable `roll_area`.

```

left = 1; roll_area = left_area = values[1] * tau; out[1] = values[1];
for (right in 2:N(X)) {
    // Expand interval on right end
    roll_area = roll_area + values[right-1] * (times[right] - times[right-1]);

    // Remove truncated area on left end
    roll_area = roll_area - left_area;

    // Shrink interval on left end
    t_left_new = times[right] - tau;
    while (times[left] <= t_left_new) {
        roll_area = roll_area - values[left] * (times[left+1] - times[left]);
        left = left + 1;
    }

    // Add truncated area on left end
    left_area = values[max(1, left-1)] * (times[left] - t_left_new)
    roll_area = roll_area + left_area;

    // Save SMA value for current time window
    out[right] = roll_area / tau;
}

```

SMA Algorithm

It is worth noting that the value of `left_area`, calculated towards the end of the loop's body, is reused in the next iteration towards the top of the loop's body.

2.3 SMA_{lin}

The simple moving average SMA_{lin} can be calculated like the SMA, except that areas entering and leaving the rolling time window are now trapezoids instead of rectangles, see Figure 2. To this end, we define a helper function that calculates the area of the trapezoid with coordinates of the corners $(x_2, 0)$, (x_2, y_2) , $(x_3, 0)$, and (x_3, y_3) , where y_2 is obtained by linear interpolation of (x_1, y_1) and (x_3, y_3) evaluated at x_2 .

```
trapezoid = function(x1, x2, x3, y1, y3) {
  if (x2 == x3 or x2 < x1)
    return (x3 - x2) * y1;
  else {
    weight = (x3 - x2) / (x3 - x1);
    y2 = y1 * weight + y3 * (1 - weight);
    return (x3 - x2) * (y2 + y3) / 2;
  }
}
```

SMA_{lin} helper function - Trapezoid Area

The second and third line the function treat the two special cases of (i) a trapezoid with zero area, and (ii) the first observation has not yet left the rolling time window. The pseudo-code for the main part of the algorithm is as follows:

```
left = 1; roll_area = left_area = values[1] * tau; out[1] = values[1];
for (right in 2:N(X)) {
  // Expand interval on right end
  roll_area = roll_area + (values[right-1] + values[right])/2 *
    (times[right] - times[right-1]);

  // Remove truncated area on left end
  roll_area = roll_area - left_area;

  // Shrink interval on left end
  t_left_new = times[right] - tau;
  while (times[left] <= t_left_new) {
    roll_area = roll_area - (values[left] + values[left+1]) / 2 *
      (times[left+1] - times[left]);
    left = left + 1;
  }

  // Add truncated area on left end
  left_area = trapezoid(times[max(1, left-1)], t_left_new, times[left],
    values[max(1, left-1)], values[left]);
  roll_area = roll_area + left_area;

  // Save SMA value for current time window
  out[right] = roll_area / tau;
}
```

SMA_{lin} Algorithm

As before, the value of `left_area`, calculated towards the end of the loop's body, is reused in the next iteration towards the top of the loop's body.

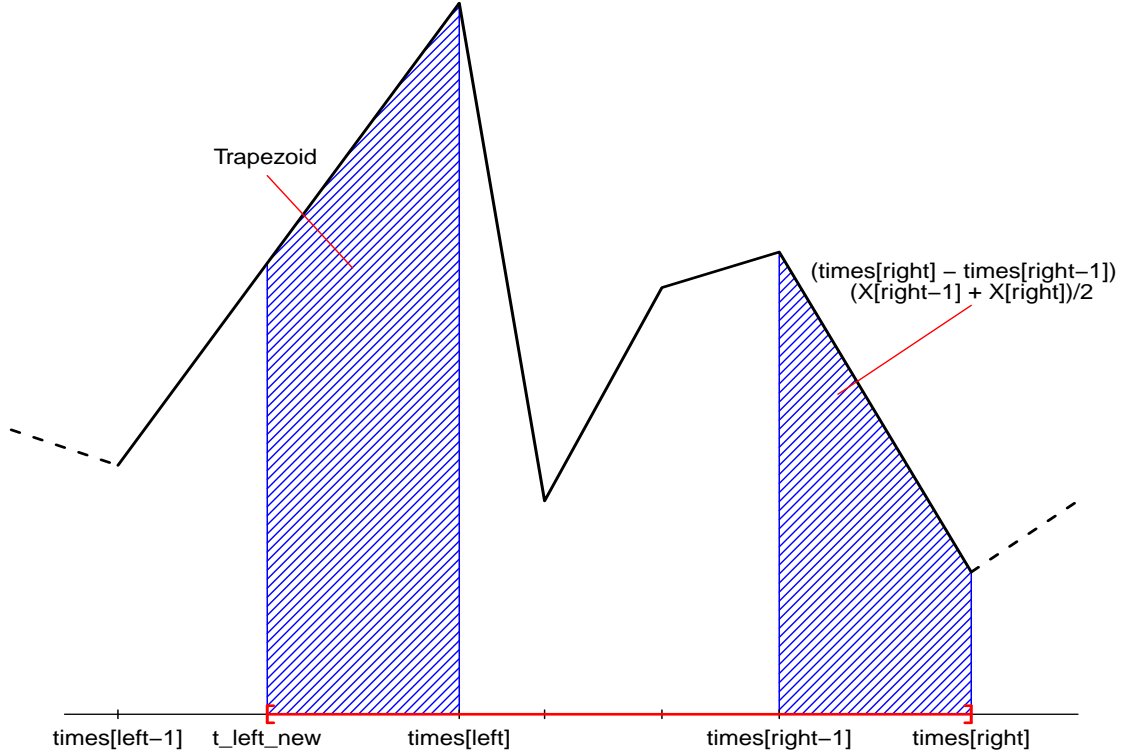


Figure 2: Areas involved in the calculation of the simple moving average $\text{SMA}_{\text{lin}}(X, \tau)$. The piecewise-linear function is the sample path of a time series X with linear interpolation. The shaded area on the right-hand side has to be added to the variable `roll_area` in order to expand the rolling time window to the right. Depending on the position in the algorithm, the shaded area on the left-hand side has to be either added or subtracted from the variable `roll_area`.

3 Exponential Moving Averages

Definition 2. For an unevenly-spaced time series X , we define three versions of the exponential moving average (EMA) of length $\tau > 0$. For $t \in \{t_1, \dots, t_{N(X)}\}$,

- (i) $\text{EMA}(X, \tau)_t = \frac{1}{\tau} \int_0^\infty X[t-s] e^{-s/\tau} ds,$
- (ii) $\text{EMA}_{\text{lin}}(X, \tau)_t = \frac{1}{\tau} \int_0^\infty X[t-s]_{\text{lin}} e^{-s/\tau} ds,$
- (iii) $\text{EMA}_{\text{eq}}(X, \tau)_t = \begin{cases} X_{t_1}, & \text{if } t = t_1 \\ (1 - e^{-\Delta t_n/\tau}) X_{t_n} + e^{-\Delta t_n/\tau} \text{EMA}_{\text{eq}}(X, \tau)_{t_{n-1}}, & \text{if } t = t_n > t_1 \end{cases}$

where in all three cases the observation times of the input and output time series are identical.

The first EMA uses last-point sampling, the second EMA uses linear interpolation, and the last EMA corresponds most widely used version for equally spaced data. See [Eckner \(2011\)](#), Section 7.2 for a theoretical analysis of these moving averages for unevenly-spaced data.

3.1 EMA_{eq}

By construction, the EMA_{eq} can be calculated recursively:

```
out[1] = values[1];
for (j in 2:N(X)) {
  w = exp(-tau * (times[j] - times[j-1]));
  out[j] = out[j-1] * w + values[j] * (1-w);
}
```

EMA_{eq} Algorithm

3.2 EMA

It is easy to show that Definition 2 (i) is equivalent to

$$\text{EMA}(X, \tau)_t = \begin{cases} X_{t_1}, & \text{if } t = t_1 \\ (1 - e^{-\Delta t_n / \tau}) X_{t_{n-1}} + e^{-\Delta t_n / \tau} \text{EMA}(X, \tau)_{t_{n-1}}, & \text{if } t = t_n > t_1 \end{cases}$$

for $t \in T(X)$, which is the EMA_{eq} of the original time series with shifted observations. Hence, the $\text{EMA}(X, \tau)$ can be calculated recursively as well:

```
out[1] = values[1];
for (j in 2:N(X)) {
  w = exp(-tau * (times[j] - times[j-1]));
  out[j] = out[j-1] * w + values[j-1] * (1-w);
}
```

EMA Algorithm

3.3 EMA_{lin}

The EMA_{lin} can also be calculated recursively, see [Eckner \(2011\)](#) for a derivation.

```
out[1] = values[1];
for (j in 2:N(X)) {
  tmp = (times[j] - times[j-1]) / tau;
  w = exp(-tmp);
  w2 = (1 - w) / tmp;
  out[j] = out[j-1] * w + values[j] * (1 - w2) + values[j-1] * (w2 - w);
}
```

EMA_{lin} Algorithm

4 Other Rolling Operators

This section presents efficient algorithms for rolling time series operators that are not moving averages, but nevertheless have a similar structure.

4.1 Rolling Sum and Number of Observations

The algorithm for the SMA_{eq} can be easily modified to calculate the total number and sum of observation values in a rolling time window of length τ .

```

left = 1;
for (right in 1:N(X)) {
  while (times[left] <= times[right] - tau)
    left = left + 1;
  out[right] = right - left + 1;
}

```

Rolling Number of Observations

```

left = 1; roll_sum = 0;
for (right in 1:N(X)) {
  roll_sum = roll_sum + X[right];
  while (times[left] <= times[right] - tau) {
    roll_sum = roll_sum - X[left];
    left = left + 1;
  }
  out[right] = roll_sum;
}

```

Rolling Sum of Observations

By construction, for a fixed rolling window width τ , the SMA_{eq} of a time series is equal to the rolling sum divided by the rolling number of observations of the same time series.

4.2 Rolling Maximum and Minimum

Assume given a function `max_index(array, start, end)` that returns the index of the largest element in an array between and including the indices `start` and `end`. The rolling maximum in a window of length $\tau > 0$ can be calculated as follows:

```

left = 1; max_pos = 1;
for (right in 1:N(X)) {
  // Expand interval on right end
  if (values[right] >= values[max_pos])
    max_pos = right;

  // Shrink interval on left end
  while (times[left] <= times[right] - tau)
    left++;

  // Recalculate position of maximum if old maximum dropped out
  if (max_pos < left)
    max_pos = max_index(values, left, right);

  // Save maximum for current time window
  out[right] = values[max_pos];
}

```

Rolling Maximum

The algorithm for calculating the rolling minimum is very similar and therefore not given here.

Remark 2. *An alternative algorithm could use a self-balancing binary search tree to keep track of the observation values inside the rolling time window, see Remark 1. The minimum and maximum element of such a data structure can be extracted in logarithmic time, which gives a better worst- case performance than the algorithm above.*⁴

⁴The worst case of the algorithm above is attained for a strictly decreasing time series, since in this case the maximum value has to be calculated from scratch for each time window.

5 More General Operators

This section briefly discusses how the operators discussed so far can be combined and generalized.

5.1 Convolution Operators

Many of the operators discussed so far are examples of convolution operators, see [Gilles Zumbach \(2001\)](#), [Dacorogna et al. \(2001\)](#), and [Eckner \(2011\)](#). For the purpose of our discussion, the following simplified definition suffices:

Definition 3. *For a time series X and function f on $\mathbb{R} \times \mathbb{R}_+$ satisfying suitable measurability and integrability conditions, the convolution $X * f$ of X with f (with last-point sampling) is given by*

$$\begin{aligned} T(X * f) &= T(X), \\ (X * f)_t &= \int_0^\infty f(X[t-s], s) ds, \quad t \in T(X * f), \end{aligned}$$

and the convolution with linear-interpolation is given by

$$\begin{aligned} T(X *_{\text{lin}} f) &= T(X), \\ (X *_{\text{lin}} f)_t &= \int_0^\infty f(X[t-s]_{\text{lin}}, s) ds, \quad t \in T(X *_{\text{lin}} f). \end{aligned}$$

The algorithms for the simple moving averages in Section 2 can be modified to allow calculating convolutions with functions of the form

$$f(x, t) = g(x) \frac{1}{\tau} \mathbf{1}_{\{0 \leq t \leq \tau\}}, \quad (5.1)$$

where $\tau > 0$ and g is a real-valued function. For example, $g(x) = x^k$ for $k \in \mathbb{N}$ allows to calculate the rolling k -th time series moment, denoted by $m(X, \tau, k)$. Of course, $k = 1$ amounts to calculating a simple moving average. In the case of linear interpolation, however, the algorithm typically requires numerical integration of the now irregular shaded areas in Figure 2. Alternatively, the function g in (5.1) could first be applied to the series values of X , and then the simple moving average SMA_{eq} , SMA , or SMA_{lin} is applied to the transformed time series.

5.2 Combining Operators

More complex time series operators, if constructed from the operators discussed so far, can also be efficiently calculated. For example, the rolling variance of a time series over a time horizon τ can be calculated as

$$\begin{aligned} \sigma^2(X, \tau) &= \text{SMA}(X^2, \tau) - (\text{SMA}(X, \tau))^2 \\ &= m(X, \tau, 2) - m(X, \tau, 1)^2, \end{aligned}$$

where X^k for a time series X is short-hand notation for taking the k -th moment of the individual time series observations.

Similarly, the average true range (ATR), a popular robust measure of price volatility, can be constructed via

$$\begin{aligned} \text{ATR}(X, \rho, \tau) &= \text{SMA}(\text{range}(X, \rho), \tau) \\ &= \text{SMA}(\text{roll_max}(X, \rho), \tau) - \text{SMA}(\text{roll_min}(X, \rho), \tau), \end{aligned}$$

where $\rho > 0$ is the range horizon, and $\tau > \rho$ is the smoothing horizon.

Finally, convolutions $X * f$ with densities of the form $f(x, t) = g(x)h(t)$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ are reasonably smooth functions, can be approximated by a linear combination of simple moving averages of the transformed time series $g(X)$, where

$$\begin{aligned} T(g(X)) &= T(X), \\ (g(X))_t &= g(X_t), \quad t \in T(g(X)). \end{aligned}$$

6 Conclusion

We have shown how to efficiently calculate a wide range of time series operators for unevenly-spaced data, and provided an implementation in the programming language C. These operators in turn may serve as building blocks for more complicated operators.

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